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BETA EXPONENTIATED INVERSE RAYLEIGH DISTRIBUTION: STATISTICAL PROPERTIES, BAYESIAN, AND NON-BAYESIAN ESTIMATION WITH APPLICATION

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Abstract

In this paper, a new distribution is proposed called beta exponentiated inverse Rayleigh (BEIR). Some of its statistical properties such as quantile function, order statistics, moments, inverse moments, moment generating function and Renyi entropy are derived and discussed. Maximum likelihood and Bayesian methods are used to estimate the model parameters. Monte-Carlo simulation study is carried out to examine the bias and mean square error of maximum likelihood and Bayesian estimators. Finally, real data sets are used to illustrate the importance of the new distribution.

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1. Introduction

Inverse Rayleigh distribution was introduced by Trayer [22] to model reliability and survival data sets. Voda [24] discussed inverse Rayleigh (IR) distribution, its properties and ML estimator of the parameter. Gharraph [7] provided a closed-form expression for the mean, harmonic mean, geometric mean, mode, and the median of this distribution. The beta inverse Rayleigh distribution (BIR) is a special case of the beta Frechet (BF) distribution, which was introduced by Nadarajah and Gupta [15]. Hassan and Parviz [25] estimated the parameters of the generalized exponential distribution using grouped data using classical and Bayesian estimation methods. Leao et al. [13] studied beta inverse Rayleigh distribution and Ahmad and Ahmed [1] introduced a generalization of the inverse Rayleigh distribution. Modified inverse Rayleigh distribution has been studied by Khan [20]. Rehman and Sajjad [19] studied exponentiated inverse Rayleigh distribution, and Khan and King [10] studied transmuted modified inverse Rayleigh distribution. Ul Haq [23] introduced transmuted exponentiated inverse Rayleigh distribution. Some new distributions have been introduced using a class of beta generalized distributions. For example, beta normal distribution (BN) was introduced by Eugene et al. [3]. General expressions for the moments of the BN distribution were derived by Gupta and Nadarajah [5]. Beta Weibull distribution was introduced by Famoye et al. [4]. Beta exponential distribution and its various properties were discussed by Nadarajah and Kotz [16]. Beta generalized exponential distribution was proposed by Hassan and Parviz [25]. Beta Weibull-geometric distribution was introduced by Bidram et al. [2]. Beta Kumaraswamy distribution was introduced by Handique et al. [8]. Beta exponential Pareto distribution, its various properties and estimation of the parameters were derived by Rashwan and Kamel [18].

The cumulative distribution function (cdf) of the exponentiated inverse Rayleigh (EIR) distribution is defined by

$$G(X) = 1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right]^{\alpha}, \quad x > 0, \sigma > 0, \alpha > 0. \tag{1}$$

The EIR density function can be written as

$$g(x) = \frac{2\alpha\sigma^2}{x^3} e^{-\left(\frac{\sigma}{x}\right)^2} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1}, \quad x > 0, \sigma > 0, \alpha > 0, \tag{2}$$

where σ is the scale parameter and α is the shape parameter.

Eugene et al. [3] defined a class of beta generalized distributions from an arbitrary baseline, $G(x)$, by

$$F(x) = \frac{1}{B(a, b)} \int_0^{G(x)} w^{a-1}(1 - w)^{b-1} dw, \quad x \succ 0, \tag{3}$$

where $a > 0$ and $b > 0$ are two additional shape parameters whose role is to introduce skewness and to vary tail weight and $B(a, b) = \int_0^1 w^{a-1}(1 - w)^{b-1} dw$ is the beta function. The cdf $G(x)$ could be quite arbitrary and F is named to be the beta G distribution.

The probability density function corresponding to equation (3) can be written in the following form:

$$f(x) = \frac{1}{B(a, b)} (G(x))^{a-1}(1 - G(x))^{b-1} g(x), \tag{4}$$

where $g(x) = \frac{dG(x)}{dx}$ is the pdf of the parent distribution.

In this article, we proposed a new distribution called the beta exponentiated inverse Rayleigh distribution (BEIRD) that includes inverse Rayleigh, beta inverse Rayleigh and exponentiated inverse Rayleigh distributions.

This paper is organized as follows: BEIRD is presented in Section 2. In Section 3, some statistical properties of the BEIR distribution are derived. In Section 4, parameters are estimated using the maximum likelihood and Bayesian methods. Moreover, a simulation study is performed to measure the efficiency of the two methods. In Section 5, an application to real data shows that the BEIR model fits better than six other lifetime models. Finally, the paper is concluded in Section 6.

2. The Beta Exponentiated Inverse Rayleigh Distribution

In this section, the beta exponentiated inverse Rayleigh distribution will be defined by taking

$$F(x) = \frac{1}{B(a, b)} \int_0^{1-\left[1-e^{-\left(\frac{\sigma}{x}\right)^2}\right]^{\alpha}} w^{a-1}(1-w)^{b-1} dw, \; a, b, \alpha, \sigma > 0, \; x > 0, \quad (5)$$

and the corresponding pdf for $F(x)$ takes the form:

$$f(x) = \frac{2\alpha\sigma^2 x^{-3}}{B(a, b)} e^{-\left(\frac{\sigma}{x}\right)^2} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha b - 1} \left(1 - \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha}\right)^{a-1}. \quad (6)$$

Note that the BEIRD involves some well-known distributions as special cases:

- when $a = b = 1$, the BEIR distribution in equation (6) reduces to the EIR distribution with parameters α and σ,

- when $a = b = \alpha = 1$, the BEIR distribution becomes the inverse Rayleigh distribution with parameter σ,

- when $\alpha = 1$, the BEIR distribution will be beta inverse Rayleigh distribution with parameters σ, a and b.

Here, simple expansions for the cdf and pdf of the BEIRD will be derived. It depends on whether the parameter b or a is a real non-integer or an integer. We use the following generalized binomial expansion:

$$(1 - Z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b - j)} Z^j, \quad |Z| < 1. \tag{7}$$

If b is a real non-integer and $\Gamma(\cdot)$ is the gamma function, applying equation (7) in equation (5), the cdf of BEIRD can be written as

$$F(x) = \frac{1}{B(a, b)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b - m)m!} \int_0^{1-\left[1-e^{-\left(\frac{\sigma}{x}\right)^2}\right]^{\alpha}} w^{\alpha+m-1} dw$$

$$= \frac{1}{B(a, b)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b - m)m!(a + m)} \left(1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right]^{\alpha}\right)^{(a+m)}. \tag{8}$$

By using equation (7) in equation (6), if a is a real non-integer, then the pdf of BEIRD can be written as:

$$f(x) = \frac{\alpha}{B(a, b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(a)\Gamma(\alpha(b + i)) g_{IR}(x; \sigma(j + 1))}{i!\, j!\Gamma(a - i)\Gamma(\alpha(b + i) - j)(j + 1)},$$

where $g_{IR}(x; \sigma(j + 1))$ is the pdf of inverse Rayleigh distribution with parameter $\sigma(j + 1)$.

Let

$$w_{ij} = \frac{\alpha(-1)^{i+j} \Gamma(a)\Gamma(\alpha(b + i))}{B(a, b)i!j!\Gamma(a - i)\Gamma(\alpha(b + i) - j)(j + 1)}.$$

Then

$$f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} g_{IR}(x; \sigma(j + 1)). \tag{9}$$

For real integers a and b, the cdf and pdf of BEIRD are expressed as follows:

$$F(x) = \frac{1}{B(a, b)} \sum_{m=0}^{b-1} \frac{(-1)^m \Gamma(b)}{\Gamma(b-m) m! (a+m)} \left(1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^2} \right]^{\alpha} \right)^{(a+m)} \tag{10}$$

and

$$f(x) = \frac{\alpha}{B(a, b)} \sum_{i=0}^{a-1} \sum_{j=0}^{\alpha(b+i)-1} \frac{(-1)^{i+j} \Gamma(a) \Gamma(\alpha(b+i)) g_{IR}(x; \sigma(j+1))}{i! j! \Gamma(a-i) \Gamma(\alpha(b+i) - j)(j+1)}. \tag{11}$$

If α is an integer, then the sum in equation (10) simply stops at $(b-1)$, and in equation (11) stops at $(\alpha(b+i)-1)$ and $(a-1)$, respectively.

The reliability function $S(x)$, hazard rate function $h(x)$, reversed hazard rate function $r(x)$, and the cumulative hazard function $H(x)$ of BEIRD are, respectively, given by:

$$S(x) = 1 - \frac{1}{B(a, b)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b-m) m! (a+m)} \left(1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^2} \right]^{\alpha} \right)^{(a+m)},$$

$$h(x) = \frac{2\alpha\sigma^2 x^{-3} e^{-\left(\frac{\sigma}{x}\right)^2} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2} \right)^{\alpha b - 1} \left(1 - \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2} \right)^{\alpha} \right)^{a-1}}{B(a, b) - \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b-m) m! (a+m)} \left(1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^2} \right]^{\alpha} \right)^{(a+m)}},$$

$$r(x) = \frac{2\alpha\sigma^2 x^{-3}\left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha b - 1}}{\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b-m)m!(a+m)}\left(1 - \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha}\right)^{1+m}}$$

and

$$H(x) = -\ln\left(1 - \frac{1}{B(a, b)}\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b-m)m!(a+m)}\left(1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right]^{\alpha}\right)^{(a+m)}\right).$$

If a random variable x has BEIRD with parameters a, b, α and σ, then we write $x \sim BEIR(a, b, \alpha, \sigma)$.

3. Statistical Properties

In this section, we introduced some of statistical properties of the BEIRD, specifically, quantile regression, order statistics, moments, inverse moments, moment generating function and Renyi entropy.

3.1. Quantile function

The quantile function of a random variable x distributed according to BEIR can be obtained by inverting equation (8) as follows:

$$Q(u) = \frac{-\sigma}{\sqrt{In\left[1 - \left(1 - \left(\frac{u}{y}\right)^{1/a+m}\right)^{1/\alpha}\right]}},$$

where $u = F(x)$ and $y = \frac{1}{B(a, b)}\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b)}{\Gamma(b-m)m!(a+m)}$.

By using $Q(u)$, we can obtain the first quartile Q_1, the median Q_2, and the third quartile Q_3 of the BEIR distribution by replacing u with values 0.25, 0.50 and 0.75, respectively.

One of the original skewness measures indicated in Bowley's skewness (*sk*), (Kenny and Keeping [12]), is defined as

$$sk = \frac{Q(0.75) - 2Q(0.50) + Q(0.25)}{Q(0.75) - Q(0.25)},$$

and the Moor's kurtosis (*ku*), (Moors [14]), which is based on octiles can be defined as

$$ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

where $Q(\cdot)$ represents the quantile function. The measures *sk* and *ku* are less sensitive to outlier values and they exist even for distributions without moments.

3.2. Order statistics

In this subsection, we consider order statistics which plays an important role in many applications such as quality control and reliability. The density function of the *i*th order statistics $X_{i:n}$ say $f_{i:n}(x)$ for $i = 1, 2, \ldots, n$ in a random sample of size *n* from the BEIRD is given by:

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)}[F(x)]^{i-1}[1-F(x)]^{n-i},$$

and applying the generalizing binomial series, we get

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}[F(x)]^{i+k-1}.$$

Substituting equation (8) in the above equation, we obtain

$$f_{i:n}(x) = \frac{f(x)}{B(i,\, n-i+1)} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\Gamma(b)}{B(a,\, b)}\right]^{i+k-1}$$

$$\times \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(b-m)m!(a+m)} \left(1 - \left[1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right]^{\alpha}\right)^{(a+m)}\right]^{i+k-1}$$

and

$$f_{i:n}(x) = \frac{f(x)}{B(i,\, n-i+1)} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\Gamma(b)}{B(a,\, b)}\right]^{i+k-1}$$

$$\times \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(b-m)m!(a+m)} \left[1 - \left[1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right]^{\alpha}\right]^{m}\right]^{i+k-1}$$

$$\times \left[1 - \left[1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right]^{\alpha}\right]^{a(i+k-1)}. \tag{12}$$

Use the expansion $\left[\sum_{m=0}^{\infty} Z_m X^m\right]^n = \sum_{m=0}^{\infty} C_{m,n} X^m$ as in Gradshteyn and Ryzhik [6], where "$C_{m,\,n} = (mz_0)^{-1} \sum_{l=1}^{m} (nl - m + l) Z_l C_{m-l,\,n}$ for $m = 1, 2, \ldots$" is the pdf of the *i*th order statistics.

By substituting in equation (12) using $f(x)$ in equation (9), the *i*th order statistics for a real non-integer $b > 0$ is given by:

$f_{i:n}(x) = f(x;\, a(i+k) + m,\, b,\, \alpha,\, \sigma)$

$$\times \sum_{k=0}^{n-i} \sum_{m=0}^{\infty} \frac{(-1)^k \binom{n-i}{k} \Gamma(b)^{i+k-1} B(a(i+k)+m,\, b) C_{m,\, i+k-1}^{(1)}}{B(i,\, n-i+1)(B(a,\, b))^{i+k}}$$

$$\times f(x;\, a(i+k)+m,\, b,\, \alpha,\, \sigma),$$

where

$$C_{m,\, i+k-1}^{(1)} = \begin{bmatrix} m^{-1}\left(\dfrac{(-1)^0}{\Gamma(b-0)(a+0)0!}\right)^{-1} \\ \sum_{l=1}^{m} [l(i+k-1)-m+l] \dfrac{(-1)^l}{l!(a+l)\Gamma(b-l)} C_{m-l,\, i+k-1} \end{bmatrix}$$

$$= \frac{a\Gamma(b)}{m} \sum_{l=1}^{m} \frac{(-1)^l (l(i+k)-m)}{l!(a+l)\Gamma(b-l)} C_{m-l,\, i+k-1},$$

and $f(x;\, a(i+k)+m,\, b,\, \alpha,\, \sigma)$ is the pdf of BEIR with parameters $a(i+k)+m,\, b,\, \alpha$ and σ.

For integer $b > 0$, the ith order statistics is given by:

$$f_{i:n}(x) = f(x;\, a(i+k)+m,\, b,\, \alpha,\, \sigma)$$

$$\times \sum_{k=0}^{n-i} \sum_{m=0}^{\infty} \frac{(-1)^k \binom{n-i}{k} B(a(i+k)+m,\, b) C_{m,\, i+k-1}^{(2)}}{B(i,\, n-i+1)(B(a,\, b))^{i+k}},$$

where

$$C_{m,\, i+k-1}^{(2)} = \begin{bmatrix} m^{-1}\left(\dfrac{(-1)^0}{(a+0)}\right)^{-1} \dbinom{b-1}{l} \\ \sum_{l=1}^{m} [l(i+k-1)-m+l] \dfrac{(-1)^l}{(a+l)} C_{m-l,\, i+k-1} \end{bmatrix}$$

$$= \frac{a}{m} \sum_{l=1}^{m} \frac{(-1)^l (l(i+k)-m) \binom{b-1}{l}}{(a+l)} C_{m-l,\, i+k-1},$$

and $f(x; a(i+k)+m, b, \alpha, \sigma)$ is the pdf of BEIR with parameters $a(i+k)+m, b, \alpha$ and σ.

3.3. Moments

The *r*th non-central moment of a random variable *x* distributed according to BEIRD is given by:

$$E(x^r) = \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \left[\sum_{i=0}^\infty \sum_{j=0}^\infty w_{ij} g_{IR}(x; \sigma(j+1))\right] dx,$$

so,

$$E(x^r) = \sum_{i=0}^\infty \sum_{j=0}^\infty w_{ij} \int_0^\infty x^r \frac{2\sigma^2}{x^3} \left[e^{-\left(\frac{\sigma}{x}\right)^2} \right]^{j+1} dx$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty w_{ij} \int_0^\infty 2\sigma^2 x^{r-3} e^{-(j+1)\left(\frac{\sigma}{x}\right)^2} dx. \tag{13}$$

Let

$$t = (j+1)\left(\frac{\sigma}{x}\right)^2, \text{ when } x = 0, t = \infty, \text{ and } x = \infty, t = 0, 0 < t < \infty,$$

$$x = \sigma(j+1)^{1/2} t^{-1/2}, \; dx = \frac{-\sigma}{2}(j+1)^{1/2} t^{-3/2} dt.$$

Using the above transformations in equation (13), we get

$$E(x^r) = (-1)\sum_{i=0}^\infty \sum_{j=0}^\infty w_{ij} \times \int_0^\infty 2\sigma^2 [\sigma(j+1)^{1/2} t^{-1/2}]^{r-3}$$

$$\cdot e^{-t} \frac{-\sigma}{2}(j+1)^{1/2} t^{-3/2} dt,$$

and then

$$E(x^r) = \sum_{i=0}^\infty \sum_{j=0}^\infty w_{ij} \int_0^\infty \sigma^r (j+1)^{\frac{r}{2}-1} t^{\frac{-r}{2}} e^{-t} dt,$$

so,

$$E(x^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} \sigma^r (j+1)^{\frac{r}{2}-1} \int_0^{\infty} t^{\left(1-\frac{r}{2}\right)-1} e^{-t} dt.$$

Hence, if $a > 0,$ then $E(x^r)$ of BEIR distribution is

$$E(x^r) = \sigma^r \Gamma\left(1 - \frac{r}{2}\right) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} (j+1)^{\frac{r}{2}-1}, \quad r \leq 1. \tag{14}$$

The above equation is valid only for $r \leq 1.$ So, the only mean of the BEIR distribution is obtained by putting $r = 1$ in equation (14) as follows:

$$E(x) = \sigma\pi \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} (j+1)^{\frac{-1}{2}}.$$

Equation (13) can be used to obtain moments and inverse moments when $r = -r.$ It is observed that the higher moments of order 2, 3 and 4 for BEIR distribution do not exist but the inverse moment of any order exists.

3.4. Inverse moments

The rth inverse moment of the BEIR distribution is obtained by:

$$E\left(\frac{1}{x}\right)^r = \int_0^{\infty} x^{-r} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} g_{IR}(x;\, \sigma(j+1)) dx,$$

so,

$$E\left(\frac{1}{x}\right)^r = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} \int_0^{\infty} x^{-r} \frac{2\sigma^2}{x^3} \left[e^{-\left(\frac{\sigma}{x}\right)^2} \right]^{j+1} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij} \int_0^{\infty} 2\sigma^2 x^{-r-3} e^{-(j+1)\left(\frac{\sigma}{x}\right)^2} dx.$$

Let

$$t = (j+1)\left(\frac{\sigma}{x}\right)^2, \text{ when } x = 0,\ t = \infty, \text{ and } x = \infty,\ t = 0,\ 0 < t < \infty,$$

$$x = \sigma(j+1)^{1/2} t^{-1/2},\ dx = \frac{-\sigma}{2}(j+1)^{1/2} t^{-3/2} dt.$$

Using the above transformations, we get:

$$E\left(\frac{1}{x}\right)^r = (-1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}$$

$$\times \int_0^{\infty} 2\sigma^2 [\sigma(j+1)^{1/2} t^{-1/2}]^{-r-3} e^{-t} \frac{-\sigma}{2}(j+1)^{1/2} t^{-3/2} dt,$$

and then

$$E\left(\frac{1}{x}\right)^r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij} \int_0^{\infty} \sigma^{-r}(j+1)^{-r-2} t^{r/2} e^{-t} dt,$$

so,

$$E\left(\frac{1}{x}\right)^r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}\sigma^{-r}(j+1)^{-r-2} \int_0^{\infty} t^{\left(1+\frac{r}{2}\right)-1} e^{-t} dt.$$

Hence, if $a > 0$, then $E\left(\frac{1}{x}\right)^r$ of BEIRD is

$$E\left(\frac{1}{x}\right)^r = \sigma^{-r}\Gamma\left(1+\frac{r}{2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}(j+1)^{-r-2}.$$

The four inverse moments are obtained by putting $r = 1, 2, 3, 4$ in the above equation:

$$E\left(\frac{1}{x}\right) = \frac{1}{2\sigma}\sqrt{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}(j+1)^{-3},$$

$$E\left(\frac{1}{x^2}\right) = \frac{1}{\sigma^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}(j+1)^{-4},$$

$$E\left(\frac{1}{x^3}\right) = \frac{3}{4\sigma^3}\sqrt{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}(j+1)^{-5},$$

$$E\left(\frac{1}{x^4}\right) = \frac{2}{\sigma^4}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}(j+1)^{-6}.$$

3.5. Moment generating function

The moment generating function (mgf) of X of the BEIRD, $M_x(t)$, can be obtained as

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx,$$

by using Taylor's series expansion, we get

$$M_x(t) = \int_0^{\infty}\left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right) f(x)dx,$$

so,

$$M_x(t) = \int_0^{\infty}\sum_{r=0}^{\infty}\frac{(tx)^r}{r!} f(x)dx,$$

and then

$$M_x(t) = E(x^r)\sum_{r=0}^{\infty}\frac{t^r}{r!}. \tag{15}$$

Hence, the moment generating function of BEIR distribution is obtained by using equation (14) in (15) as:

$$M_x(t) = \sigma^r\Gamma\left(1 - \frac{r}{2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r w_{ij}(j+1)^{\frac{r}{2}-1}}{r!}, \quad r \le 1.$$

In the same way, the factorial moment generating function of the BEIR distribution becomes:

$$M_x(\ln t) = E(e^{x\ln t}) = \sigma^r\Gamma\left(1-\frac{r}{2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\ln t)^r w_{ij}(j+1)^{\frac{r}{2}-1}}{r!},$$

$$r \le 1,$$

and the characteristic function of BEIR distribution is given by:

$$\Phi_x(t) = E(e^{itx}) = \sigma^r\Gamma\left(1-\frac{r}{2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r w_{ij}(j+1)^{\frac{r}{2}-1}}{r!}, \; r \le 1,$$

where $i = \sqrt{-1}$ is the unit imaginary number.

3.6. Renyi entropy

The entropy of random variable x with density function $f(x)$ is a measure of variation of the uncertainty, as in Song [21], Renyi entropy is given by:

$$I_{x:R}(q) = \frac{1}{1-q}\ln(I_x(q)),$$

where $I_x(q) = \int f^q(x)dx,\; q > 0,\; q \ne 1.$

For a random variable x distributed as BEIR and by using equation (9), we have:

$$I_x(q) = \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}\right)^q \int_0^{\infty}\left(\frac{2\sigma^2}{x^2}\left[e^{-\left(\frac{\sigma}{x}\right)^2}\right]^{j+1}\right)^q dx,$$

and then

$$I_x(q) = \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}\right)^q (2\sigma)^q \int_0^{\infty} x^{-3q} e^{-q(j+1)\left(\frac{\sigma}{x}\right)^2} dx.$$

Let

$$t = q(1 + j)\left(\frac{\sigma}{x}\right)^2, \text{ when } x = 0,\ t = \infty, \text{ and } x = \infty,\ t = 0,\ 0 < t < \infty,$$

$$x = \sigma(q(1 + j))^{1/2} t^{-1/2},\ dx = \frac{-\sigma}{2}(q(1 + j))^{1/2} t^{-3/2} dt.$$

By using the above transformation, we have:

$$I_x(q) = \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}\right)^q 2^{q-1}\sigma^{1-2q}(q(j+1))^{\frac{1-3q}{2}} \int_0^{\infty} t^{\frac{3q-3}{2}+1-1} e^{-t} dt,$$

and then

$$I_x(q) = \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}\right)^q 2^{q-1}\sigma^{1-2q}(q(j+1))^{\frac{1-3q}{2}} \Gamma\left(\frac{3q-1}{2}\right).$$

Hence, the Renyi entropy becomes:

$$I_{x:R}(q) = \frac{q}{1-q}\ln\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}\right) - \ln 2 + \ln\sigma$$

$$+ \left(\frac{1-3q}{2-2q}\right)\ln(q(j+1)) + \ln\Gamma\left(\frac{3q-1}{2}\right).$$

4. Estimation

In this section, the parameters of the proposed BEIRD will be estimated using the maximum likelihood estimation method and Bayesian estimation method as follows:

4.1. Maximum likelihood estimation and its simulation

The maximum likelihood estimation method of the unknown vector of parameters ϕ, where $\phi = (a, b, \alpha, \sigma)$, will be used to estimate the unknown parameters of the BEIRD. Let $x_1, x_2, \ldots, x_n$ be an independent random sample of size n from the BEIRD with parameters a, b, α and σ. Then the likelihood function, L, of BEIRD is given by:

$$L(\phi) = \left[\prod_{i=1}^{n} f(x;\, a,\, b,\, \theta,\, \alpha,\, \lambda)\right] = \left(\frac{2\alpha\sigma^2}{\beta(a,\, b)}\right)^n \prod_{i=1}^{n} x_i^{-3} \prod_{i=1}^{n} e^{-\left(\frac{\sigma}{x_i}\right)^2}$$

$$\times \prod_{i=1}^{n} \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha b - 1} \prod_{i=1}^{n} \left(1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}\right)^{a-1}, \qquad (16)$$

and the logarithm likelihood function $\ln L$ for the vector of parameters $\phi = (a,\, b,\, \alpha,\, \sigma)^T$ can be expressed as

$$L(\phi) = n \ln 2 + n \ln \sigma^2 + n \ln \alpha - n \ln B(a,\, b) - 3\sum_{i=1}^{n} \ln x_i - \sigma^2 \sum_{i=1}^{n} x_i^{-2}$$

$$+ (\alpha b - 1)\sum_{i=1}^{n} \ln\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)$$

$$+ (a - 1)\sum_{i=1}^{n} \ln\left(1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}\right). \qquad (17)$$

The score vector is:

$$U(\phi) = \left[\frac{\partial L(\phi)}{\partial a},\, \frac{\partial L(\phi)}{\partial b},\, \frac{\partial L(\phi)}{\partial \alpha},\, \frac{\partial L(\phi)}{\partial \sigma}\right]^T,$$

where the components corresponding to the parameters in ϕ are calculated by differentiating equation (17) as follows:

$$\frac{\partial L}{\partial a} = n(\psi(a,\, b) - \psi(a)) + \sum_{i=1}^{n} \ln\left(1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}\right), \qquad (18)$$

$$\frac{\partial L}{\partial b} = n(\psi(a, b) - \psi(b)) + \alpha\sum_{i=1}^{n} \ln\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right), \tag{19}$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + b\sum_{i=1}^{n} \ln\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)$$

$$+ (a-1)\sum_{i=1}^{n} \frac{\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha} \ln\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)}{1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}}, \tag{20}$$

$$\frac{\partial L}{\partial \sigma} = \frac{2n}{\sigma} - 2\sigma\sum_{i=1}^{n} x_i^{-2} + 2\sigma(\alpha b - 1)\sum_{i=1}^{n} \frac{e^{-\left(\frac{\sigma}{x_i}\right)^2} x_i^{-2}}{1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}}$$

$$+ 2\sigma\alpha(a-1)\sum_{i=1}^{n} \frac{\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha - 1} e^{-\left(\frac{\sigma}{x_i}\right)^2} x_i^{-2}}{1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}}, \tag{21}$$

where $\psi(\cdot)$ is the digamma function which is the derivative of $\ln\Gamma(\cdot)$, where $\Gamma(\cdot)$ is the gamma function. We can obtain the estimates of the unknown parameters by setting the score vector to zero and solving them using numerical iteration such as Newton-Raphson algorithm.

Now, we study the performance of the MLE with respect to sample size *n* using simulation. To conduct the simulation study, we follow the following steps:

We generate 1000 samples of size $n = 20, 50, 100, 200$ from BEIRD (2, 0.5, 0.75, 1.5) and compute the MLEs for the 1000 samples, and compute the biases and mean-squared errors (MSEs). The results are listed in Table 1, where

$$\text{MSE} = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\phi}_\iota - \phi)^2, \quad \text{Bias} = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\phi}_\iota - \phi).$$

Table 1. Average of "MLE summaries, bias and MSE"

n	Parameters	Average "MLE"	Bias	MSE
20	a	2.07980	0.13982	0.08317
	b	0.48813	0.02187	0.03695
	α	0.76427	0.73106	0.05837
	σ	1.56790	0.15036	0.09734
50	a	2.06627	0.09247	0.08094
	b	0.48299	0.01501	0.02913
	α	0.77527	0.65116	0.04637
	σ	1.54977	0.11943	0.08641
100	a	2.05368	0.08477	0.07365
	b	0.48420	0.01080	0.02081
	α	0.77760	0.40657	0.03971
	σ	1.51748	0.09067	0.07006
200	a	2.06638	0.07402	0.06824
	b	0.49017	0.00983	0.00959
	α	0.75726	0.14822	0.02864
	σ	1.49795	0.05068	0.07332

From Table 1, it is noted that the magnitude of bias and MSEs always decrease as n grows. Thus, the MLE technique performed quite well for estimating the parameters.

4.2. Bayesian estimation and its simulation

In this subsection, Bayesian estimation of the unknown vector of parameters ϕ of the BEIR is considered under the squared error loss function. Assuming that the unknown parameters are independent, the Bayesian estimation for ϕ is obtained assuming the standard exponential distribution as an informative prior for each parameter, in Case 1. While Case 2 assumes a gamma prior for each one of the parameters, Case 3 deals with the non-informative prior distribution for the parameters.

Case 1 (called C1). Suppose that the prior distribution of each element of the vector of parameters $\phi = (a, b, \alpha, \sigma)$ is a standard exponential distribution. Then the joint prior density function of parameters ϕ is given by:

$$\pi_1(\phi) = e^{-a}e^{-b}e^{-\alpha}e^{-\sigma}, \tag{22}$$

where a, b, α and σ are positive.

The joint posterior density function of ϕ can be obtained from equations (16) and (22) as follows:

$$\pi_1(\phi | x) \propto L(\phi)\pi_1(\phi),$$

so,

$$\pi_1(\phi | x) \propto e^{-a}e^{-b}e^{-\alpha}e^{-\sigma}\left(\frac{2\alpha\sigma^2}{\beta(a, b)}\right)^n \prod_{i=1}^n x_i^{-3} \prod_{i=1}^n e^{-\left(\frac{\sigma}{x_i}\right)^2}$$

$$\times \prod_{i=1}^n \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha b - 1} \prod_{i=1}^n \left(1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}\right)^{a-1}. \tag{23}$$

Case 2 (called C2). Suppose that the prior distribution of each element of the vector of parameters $\phi = (a, b, \alpha, \sigma)$ is a gamma $(\delta i, 1)$ distribution; $i = 1, 2, 3, 4$. Then the joint prior density function of the vector of parameters ϕ is given by

$$\pi_2(\phi) = \frac{1}{\Gamma\delta_1} a^{\delta_1 - 1} e^{-a} \frac{1}{\Gamma\delta_2} b^{\delta_2 - 1} e^{-b} \frac{1}{\Gamma\delta_3} \alpha^{\delta_3 - 1} e^{-\alpha} \frac{1}{\Gamma\delta_4} \sigma^{\delta_4 - 1} e^{-\sigma}. \tag{24}$$

The joint posterior density function of ϕ can be obtained from equations (16) and (24) as follows:

$$\pi_2(\phi | x) \propto L(\phi)\pi_2(\phi),$$

so,

$$\pi_2(\phi | x) \propto \frac{1}{\Gamma\delta_1} a^{\delta_1 - 1} e^{-a} \frac{1}{\Gamma\delta_2} b^{\delta_2 - 1} e^{-b} \frac{1}{\Gamma\delta_3} \alpha^{\delta_3 - 1} e^{-\alpha}$$

$$\cdot \frac{1}{\Gamma\delta_4} \sigma^{\delta_4 - 1} e^{-\sigma} \left(\frac{2\alpha\sigma^2}{\beta(a, b)} \right)^n \prod_{i=1}^{n} x_i^{-3}$$

$$\times \prod_{i=1}^{n} e^{-\left(\frac{\sigma}{x_i}\right)^2} \prod_{i=1}^{n} \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2} \right)^{\alpha b - 1}$$

$$\cdot \prod_{i=1}^{n} \left(1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2} \right)^{\alpha} \right)^{a-1}. \tag{25}$$

Case 3 (called C3). Assume a non-informative distribution for each parameter. Then the joint prior density function of the vector of the parameters ϕ is given by:

$$\pi_3(\phi) \propto \frac{1}{a} \frac{1}{b} \frac{1}{\alpha} \frac{1}{\sigma}. \tag{26}$$

The joint posterior density function of ϕ can be obtained from equations (16) and (26) as follows:

$$\pi_3(\phi | x) \propto L(\phi)\pi_3(\phi),$$

so,

$$\pi_3(\phi | x) \propto \frac{1}{a}\frac{1}{b}\frac{1}{\alpha}\frac{1}{\sigma}\left(\frac{2\alpha\sigma^2}{\beta(a, b)}\right)^n \prod_{i=1}^{n} x_i^{-3} \prod_{i=1}^{n} e^{-\left(\frac{\sigma}{x_i}\right)^2}$$

$$\times \prod_{i=1}^{n}\left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha b - 1} \prod_{i=1}^{n}\left(1 - \left(1 - e^{-\left(\frac{\sigma}{x_i}\right)^2}\right)^{\alpha}\right)^{a-1}. \quad (27)$$

The conditional posterior distribution of a, b, α and σ cannot be reduced analytically to well-known distributions, and therefore, it is not possible to sample directly by standard methods. We use the Markov Chain Monte Carlo (MCMC) method named as the Metropolis Hastings sampling. For this algorithm, we propose the following steps:

Step 1. Choose the MLEs $\hat{a}, \hat{b}, \hat{\alpha}$ and $\hat{\sigma},$ as the starting values $(a^{(1)}, b^{(0)}, \alpha^{(0)}, \sigma^{(0)})$ of a, b, α and σ.

Step 2. Set $i = 1$.

Step 3. Generate $a^{(i)}, b^{(i)}, \alpha^{(i)}, \sigma^{(i)}$.

Step 4. Evaluate the acceptance probabilities η_ϕ.

Step 5. If $\eta_\phi < \phi^{(i)},$ accept the proposal points.

Step 6. If $\eta_\phi > \phi^{(i)},$ the proposal points are rejected and set $i = i + 1$.

Step 7. Repeat Steps 3-5, for all $i = 1, 2, \ldots, 1000$ (N times).

Step 8. Obtain the Bayes estimates of a, b, α and σ with respect to the square error loss function as

$$E(a \mid x) = \frac{1}{N-m} \sum_{i=m+1}^{N} a_i,$$

$$E(b \mid x) = \frac{1}{N-m} \sum_{i=m+1}^{N} b_i,$$

$$E(\alpha \mid x) = \frac{1}{N-m} \sum_{i=m+1}^{N} \alpha_i,$$

$$E(\sigma \mid x) = \frac{1}{N-m} \sum_{i=m+1}^{N} \sigma_i,$$

where m is the burn-in period, $\eta_\phi = \min\left(1, \frac{\pi_c(\phi^i \mid x)}{\pi_c(\phi^{i-1} \mid x)}\right)$, and $\phi = (a, b, \alpha, \sigma)$.

To compute the posterior summaries, the computations regarding the comparisons between the considered three cases (C1, C2 and C3) are performed assuming different sample sizes. For a given vector of parameters $\phi = (a, b, \alpha, \sigma)$, we generated 10,000 MCMC samples and used 2002 as the burn-in period to have stable posterior summaries. The resulting study is tabulated in Tables 2, 3 and 4.

Table 2. Posterior summaries for the BEIR distribution for C1

Prior	n	Parameters	Mean	Biases	MSE
C1	20	$\hat{a}$	2.06271	0.09749	0.00971
		$\hat{b}$	0.48159	0.01845	0.00169
		$\hat{\alpha}$	0.77721	0.06084	0.00475
		$\hat{\sigma}$	1.55953	0.08610	0.00814
	50	$\hat{a}$	2.07079	0.09507	0.00936
		$\hat{b}$	0.48509	0.01494	0.00133
		$\hat{\alpha}$	0.77311	0.05786	0.00465
		$\hat{\sigma}$	1.54868	0.08079	0.00760

	100	$\hat{a}$	2.06297	0.09183	0.00885
		$\hat{b}$	0.48448	0.01354	0.00129
		$\hat{\alpha}$	0.75947	0.05146	0.00379
		$\hat{\sigma}$	1.51633	0.07503	0.00680
	200	$\hat{a}$	2.04944	0.08298	0.00772
		$\hat{b}$	0.48664	0.01038	0.00109
		$\hat{\alpha}$	0.75308	0.04406	0.00294
		$\hat{\sigma}$	1.48404	0.07667	0.00674

A program code has been designed using R statistical package to solve the integral of equations (23), (25) and (27), to obtain the estimates $\hat{\phi} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\sigma})$ of $\phi = (a, b, \alpha, \sigma)$ in all three previous cases. Tables 2, 3 and 4 show the estimate of mean, bias, and mean squared error (MSE) for the three considered cases (C1, C2 and C3).

Table 3. Posterior summaries for the BEIR distribution for C2

Prior	n	Parameters	Mean	Biases	MSE
C2	20	$\hat{a}$	2.07979	0.09801	0.00972
		$\hat{b}$	0.48811	0.01191	0.00268
		$\hat{\alpha}$	0.77427	0.06105	0.00498
		$\hat{\sigma}$	1.56788	0.08828	0.00855
	50	$\hat{a}$	2.06626	0.09245	0.00907
		$\hat{b}$	0.48299	0.01704	0.00151
		$\hat{\alpha}$	0.76527	0.06015	0.00483
		$\hat{\sigma}$	1.54976	0.08200	0.00764
	100	$\hat{a}$	2.05368	0.09079	0.00831
		$\hat{b}$	0.48421	0.01583	0.00140
		$\hat{\alpha}$	0.75761	0.05657	0.00433
		$\hat{\sigma}$	1.51748	0.07658	0.00688
	200	$\hat{a}$	2.06640	0.08448	0.00798
		$\hat{b}$	0.49017	0.00985	0.00080
		$\hat{\alpha}$	0.75726	0.04822	0.00344
		$\hat{\sigma}$	1.49708	0.06433	0.00521

Tables 2, 3 and 4 show that the MSEs of all parameters are decreasing when the sample size is increasing. However, the MSEs of all parameters are very large, when considering Bayesian estimation based on the non-informative prior (C3). That is, in general, the Bayesian estimation based on the informative priors provides smaller MSE than the Bayesian estimation based on the non-informative prior. For all sample sizes, the Bayesian estimation according to the standard exponential prior distributions (C1) provides the best estimate for the parameters, since their corresponding MSEs are small. Bayesian estimation gives better estimation than maximum likelihood estimation.

Table 4. Posterior summaries for the BEIR distribution for C3

Prior	n	Parameters	Mean	Biases	MSE
C3	20	$\hat{a}$	2.04921	0.09793	0.00968
		$\hat{b}$	0.47182	0.02821	0.00256
		$\hat{\alpha}$	0.77023	0.06107	0.00508
		$\hat{\sigma}$	1.55687	0.08920	0.00855
	50	$\hat{a}$	2.06550	0.09298	0.00912
		$\hat{b}$	0.47941	0.02066	0.00190
		$\hat{\alpha}$	0.77502	0.05879	0.00458
		$\hat{\sigma}$	1.54694	0.08410	0.00774
	100	$\hat{a}$	2.04768	0.08951	0.00859
		$\hat{b}$	0.47846	0.01357	0.00189
		$\hat{\alpha}$	0.76399	0.05085	0.00409
		$\hat{\sigma}$	1.51308	0.07509	0.00671
	200	$\hat{a}$	2.07396	0.08063	0.00858
		$\hat{b}$	0.49484	0.00517	0.00037
		$\hat{\alpha}$	0.76412	0.04524	0.00294
		$\hat{\sigma}$	1.49711	0.07263	0.00631

5. Data Analysis

This section contains an application of the BEIRD for a real data. The data set consists of 74 observations and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20mm (Kundu and Raqab [11]).

The data set is:

1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997 2.006 2.021 2.027 2.055 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270 2.272 2.274 2.301 2.301 2.359 2.382 2.382 2.426 2.434 2.435 2.478 2.490 2.511 2.514 2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684 2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.809 2.818 2.821 2.848 2.880 2.954 3.012 3.067 3.084 3.090 3.096 3.128 3.233 3.433 3.585 3.585.

The required numerical evaluations are implemented using Mathematica package software. We use the above real data to compare the fits of the proposed model, BEIR and those of other sub-models, i.e., exponentiated inverse Rayleigh distribution (EIR), transmuted inverse Rayleigh (TIR), odd Lindley Rayleigh (OLR), Rayleigh (R), inverse Rayleigh (IR) and area biased Rayleigh (ABR). Plots of the estimated density and expected value for data set are given in Figure 1 and the empirical $F(x)$ and $S(x)$ plots for data set are given in Figure 2.

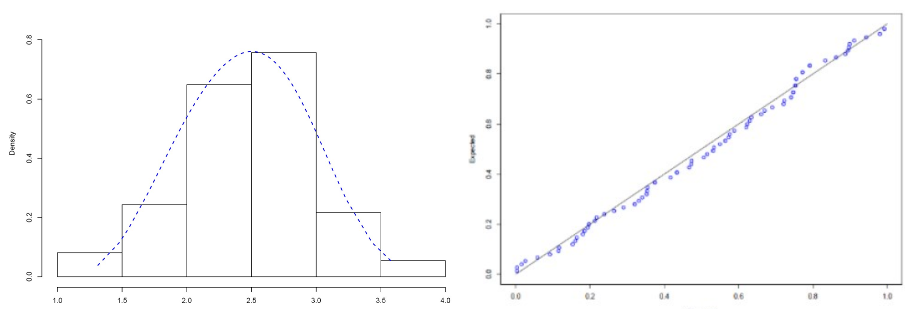

Figure 1. The PDF and expected value plots for data set.

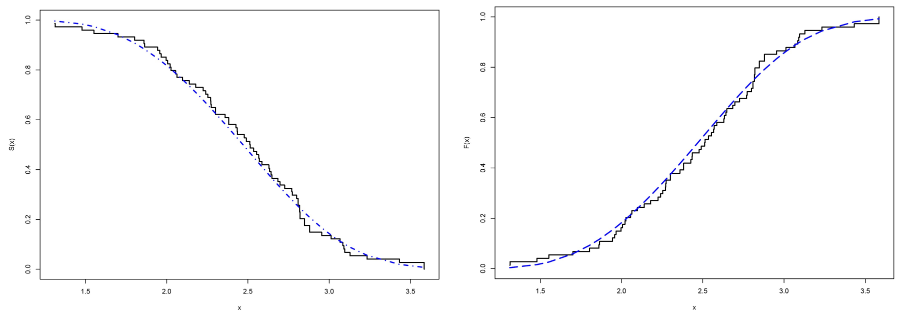

Figure 2. The empirical $S(x)$ and $F(x)$ plots for data set.

Table 5 gives the MLEs and standard error of the model parameters for BEIR, EIR, TIR, OLR, R, IR, and ABR. Table 6 lists the value of "Anderson Darling test (AD), Watson test (W), Kolmogorov Smirnov statistic (KM), the value of the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and *p*-value" for BEIR, EIR, TIR, OLR, R, IR, and ABR.

Note that $AIC = 2k - 2\ln(L)$, $BIC = k\ln(n) - 2\ln(\hat{L})$, and $HQIC = -2L_{\max} + 2k\ln(\ln(n))$, where *k* is the number of parameters, *L* is likelihood function, *n* is the number of data points, $\hat{L}$ is the maximized value of the likelihood function and $L_{\max}$ is the log-likelihood.

From Table 6, the results indicate that the BEIRD has the smaller value of AD, W, KS, AIC, BIC and HQIC. Also, BEIRD has the bigger value of *p*-value when compared to that of the EIR, TIR, OLR, R, IR and ABR models. So, the model of BEIR provides a better fit to these data and seems to be a very competitive model for these data.

Table 5. Estimated parameters with their standard errors of the BEIR model and other fitted models

Model	MLE				Standard error			
	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$	α	σ	a	b
BEIR	10.4517	6.95714	0.244047	11.2401	78.7445	1.22166	0.107638	82.1721
EIR	11.0291	4.0508	-	-	3.02972	0.221016	-	-
TIR	1	7.85254	-	-	0.665047	1.21713	-	-
OLR	0.340327	0.506879	-	-	0.0814502	0.0520208	-	-
R	-	1.78487	-	-	-	0.103743	-	-
IR	-	5.3379	-	-	-	0.620518	-	-
ABR	-	1.36209	-	-	-	0.0518716	-	-

Table 6. Goodness of fit measures of the BEIR model and other competing models

Model	AD	W	KS	AIC	BIC	HQIC	p-value
BEIR	0.365892	0.0488351	0.0595218	111.667	120.883	115.344	0.955728
EIR	0.924721	0.140274	0.0865446	115.149	119.757	116.988	0.636459
TIR	6.20894	1.08041	0.25045	155.29	159.898	157.128	0.000185916
OLR	89.2295	16.4	0.926201	116.217	120.826	120.826	0
R	13.3126	2.65435	0.339298	190.302	192.606	191.221	7.96907×10^{-8}
IR	12.3913	2.49416	0.366424	193.091	195.395	194.01	4.68758×10^{-9}
ABR	6.30935	1.12823	0.004295	144.833	147.137	145.753	0.001168

6. Conclusion

This article introduces a four parameters model, called beta exponentiated inverse Rayleigh model. The proposed distribution includes special sub-models. Some mathematical properties are derived. Parameters of BEIRD are estimated by using the maximum likelihood estimation method and Bayesian estimation method. The Bayesian estimation of the parameters under squared error loss function was considered for the BEIR (a, b, α, σ) distribution. The joint posterior distribution was introduced by using both informative and non-informative prior distributions. Based on Monte Carlo simulation study, it has been observed that the Bayesian estimates of ϕ under the assumption of three cases are the same. The proposed distribution is applied to a real data set. The BEIRD provides a better fit than several other sub-models. It has been observed that the Bayesian estimates of ϕ under the assumption of the standard exponential prior distributions have the smallest MSE, when compared to the other cases. Also, Bayesian estimation based on informative prior distributions is found better than Bayesian estimation based on non-informative prior distributions.

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